

# ON THE SIMULTANEOUS APPROXIMATION OF COEFFICIENTS OF SCHLICHT FUNCTIONS

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**ABSTRACT.** A modified version of the Hardy-Littlewood tauberian theorem is used to prove under which conditions the moduli of the coefficients  $|a(n)|/n$  of schlicht functions tend uniformly to their Hayman-indexes as  $n \rightarrow \infty$ .

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In the sequel let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , let  $\mathbb{D}$  denote the unit disk, let  $\Delta = \{z \in \mathbb{C} : |z| > 1\}$  and let  $S$  denote the set of schlicht functions that are univalent in  $\mathbb{D}$ . A function  $g : \Delta \rightarrow \mathbb{C}$ ,  $g(z) = z + \sum_{n=0}^{\infty} b_n z^{-n}$  analytic and univalent in  $\Delta$  is called a full mapping if the complement of  $g(\Delta)$  with respect to  $\mathbb{C}$  has two-dimensional Lebesgue-measure zero, the corresponding class is denoted by  $\tilde{S}$  (for further details see for instance [1], chapter 2). Suppose that  $f \in S$  is given by  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and let  $0 \leq \alpha \leq 1$  denote its Hayman-index. The Hayman-index  $\alpha$  of a schlicht function  $f \in S$  is defined by the formula  $\alpha = \lim_{r \rightarrow 1^-} (1-r)^2 M_{\infty}(r, f)$ , where  $M_{\infty}(r, f) = \max\{|f(z)| : |z| = r\}$ .  $f \in S$  is said to be of slow growth if  $\alpha = 0$  and it is said to be of maximal growth if  $\alpha > 0$ . Then Hayman's regularity theorem asserts that  $|a_n|/n \rightarrow \alpha$  as  $n \rightarrow \infty$ , however by a result of Shirokov ([2])  $(|a_n|/n)$  may converge arbitrarily slowly to  $\alpha$  as  $n \rightarrow \infty$ . So the question arises under which conditions the terms  $|a_n|/n$  converge more regularly to  $\alpha$  as  $n \rightarrow \infty$ . In order to give an answer to this question families of schlicht functions with certain properties will be considered in the sequel and the tool mainly used will be an extension of the Hardy-Littlewood tauberian theorem, which is introduced in Lemma 1. Its proof will be given here since the author isn't aware of any reference.

**Lemma 1.** (*Simultaneous tauberian approximation*). Let  $(f_m)$ ,  $f_m(z) = \sum_{k=0}^{\infty} a_k^{(m)} z^k$  denote a sequence of functions analytic in the unit disk with real coefficients, that is  $a_k^{(m)} \in \mathbb{R}$  for  $k \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$ . Furthermore for  $m \in \mathbb{N}$  let  $s_n^{(m)}$ ,  $n \in \mathbb{N}_0$  be defined by  $s_n^{(m)} = \sum_{k=0}^n a_k^{(m)}$  and suppose that

- (i) there exist constants  $0 < K < \infty$  and  $\alpha_m \in \mathbb{R}$ ,  $m \in \mathbb{N}$  such that  $\lim_{t \rightarrow 1^-} f_m(t) = \alpha_m$  and  $|\alpha_m| < K$
- (ii)  $(f_m)$  converges uniformly in  $[0, 1]$  as  $m \rightarrow \infty$
- (iii) there exists a constant  $0 < L < \infty$  such that  $s_n^{(m)} - \alpha_m < L$  for every  $m \in \mathbb{N}$ ,  $n \in \mathbb{N}_0$

Then for every  $\epsilon > 0$  there exists a  $N(\epsilon) \in \mathbb{N}$  such that

$$\left| \alpha_m - \frac{1}{n+1} \sum_{k=0}^n (n+1-k) a_k^{(m)} \right| = \left| \alpha_m - \frac{1}{n+1} \sum_{k=0}^n s_k^{(m)} \right| < \epsilon$$

whenever  $m, n > N(\epsilon)$ .

*Proof.* Let  $\epsilon > 0$  and consider  $g_m(t) = f_m(t) - \alpha_m = (1-t) \sum_{n=0}^{\infty} (s_n^{(m)} - \alpha_m) t^n$ ,  $m \in \mathbb{N}$ . Define  $\lambda : [0, 1] \rightarrow \mathbb{R}$  by

$$t\lambda(t) = \begin{cases} 0 & \text{if } 0 \leq t < e^{-1} \\ 1 & \text{if } e^{-1} \leq t \leq 1 \end{cases}.$$

Then by the Weierstrass-Approximation-Theorem (see [3], chapter 7.53) there exist polynomials  $p_\epsilon, P_\epsilon$  such that  $p_\epsilon(t) < \lambda(t) < P_\epsilon(t)$  and

$$(1) \quad \int_0^1 P_\epsilon(t) - p_\epsilon(t) dt < \epsilon L^{-1}.$$

Now let  $P_\epsilon(t) - p_\epsilon(t) = \sum_{j=0}^v d_j t^j$ ,  $d_j \in \mathbb{R}$ ,  $j = 0, \dots, v$ . Then on one hand

$$(2) \quad (1-t) \sum_{k=0}^{\infty} t^k (P_\epsilon(t^k) - p_\epsilon(t^k)) = (1-t) \sum_{j=0}^v d_j \sum_{k=0}^{\infty} (t^{j+1})^k = \sum_{j=0}^v d_j \frac{1-t}{1-t^{j+1}}$$

and on the other hand

$$(3) \quad \sum_{j=0}^v d_j \frac{1-t}{1-t^{j+1}} \rightarrow \sum_{j=0}^v \frac{d_j}{j+1} = \int_0^1 P_\epsilon(t) - p_\epsilon(t) dt$$

as  $t \rightarrow 1^-$ . Hence by (2) and (3) there exists a  $T_1(\epsilon)$  with  $0 < T_1(\epsilon) < 1$  such that

$$(4) \quad \left| (1-t) \sum_{k=0}^{\infty} t^k (P_\epsilon(t^k) - p_\epsilon(t^k)) - \int_0^1 P_\epsilon(t) - p_\epsilon(t) dt \right| < \epsilon L^{-1}$$

if  $1 \geq t > T_1(\epsilon)$ . But by (1), (4) and (iii)

$$\begin{aligned} (1-t) \sum_{k=0}^{\infty} (s_k^{(m)} - \alpha_m) (t^k \lambda(t^k) - t^k p_\epsilon(t^k)) &< (1-t) \sum_{k=0}^{\infty} L t^k (\lambda(t^k) - p_\epsilon(t^k)) \\ &< (1-t) L \sum_{k=0}^{\infty} t^k (P_\epsilon(t^k) - p_\epsilon(t^k)) < 2\epsilon \end{aligned}$$

and

$$\begin{aligned} (1-t) \sum_{k=0}^{\infty} (s_k^{(m)} - \alpha_m) (t^k P_\epsilon(t^k) - t^k \lambda(t^k)) &< (1-t) \sum_{k=0}^{\infty} L t^k (P_\epsilon(t^k) - \lambda(t^k)) \\ &< (1-t) L \sum_{k=0}^{\infty} t^k (P_\epsilon(t^k) - p_\epsilon(t^k)) < 2\epsilon \end{aligned}$$

if  $1 \geq t > T_1(\epsilon)$ . The last two inequalities imply that

$$(5) \quad (1-t) \sum_{k=0}^{\infty} c_k^{(m)} t^k P_{\epsilon}(t^k) - 2\epsilon < (1-t) \sum_{k=0}^{\infty} c_k^{(m)} t^k \lambda(t^k) < (1-t) \sum_{k=0}^{\infty} c_k^{(m)} t^k p_{\epsilon}(t^k) + 2\epsilon$$

where  $c_k^{(m)} = s_k^{(m)} - \alpha_m$ ,  $k \in \mathbb{N}_0$ ,  $m \in \mathbb{N}$  if  $1 \geq t > T_1(\epsilon)$ . The next step is to show that the terms on the left hand side and on the right hand side of (5) involving  $P_{\epsilon}$  and  $p_{\epsilon}$  are smaller than  $\epsilon$  if  $t \in (0, 1)$  is chosen large enough. This will be shown for an arbitrarily chosen polynomial  $P$ , so that it will also hold for  $P_{\epsilon}$  and  $p_{\epsilon}$ . Let  $P$  be defined by  $tP(t) = \sum_{j=1}^{\mu} b_j t^j$  and show that there exist constants  $T(\epsilon) > 0$  and  $M \in \mathbb{N}$  (dependent on  $P$  and  $\epsilon$ ) such that

$$(6) \quad (1-t) \left| \sum_{k=0}^{\infty} (s_k^{(m)} - \alpha_m) t^k P(t^k) \right| < 2\epsilon$$

whenever  $T(\epsilon) < t^{\mu} < 1$  and  $m > M$ . In order to prove (6) observe that by hypothesis the sequence  $(g_m)$  converges uniformly on  $[0, 1]$ . Hence there exists a  $M \in \mathbb{N}$  such that for  $0 \leq t \leq 1$

$$(7) \quad |g_M(t) - g_m(t)| < \min \left\{ \epsilon \left( \sum_{i=1}^{\mu} |b_i| \right)^{-1}, \epsilon \right\}$$

if  $m > M$ . But since  $g_M$  is continuous on  $[0, 1]$  by (i) and since  $g_M(1) = 0$  there exists a  $T(\epsilon) > 0$  such that

$$(8) \quad |g_M(t^j)| < \min \left\{ \epsilon \left( \sum_{i=1}^{\mu} |b_i| \right)^{-1}, \epsilon \right\}$$

if  $T(\epsilon) < t^{\mu} \leq 1$  for  $j = 1, \dots, \mu$ . Consequently by (8)

$$(9) \quad \sum_{j=1}^{\mu} |b_j| |g_M(t^j)| < \epsilon$$

and (7) implies that

$$(10) \quad \sum_{j=1}^{\mu} |b_j| |g_M(t^j) - g_m(t^j)| < \epsilon$$

whenever  $T(\epsilon) < t^{\mu} \leq 1$  and  $m > M$ . Now, let  $t \in (0, 1)$  be chosen arbitrarily, let  $y_j$  be defined by  $y_j = t^j$  for  $j = 1, \dots, \mu$  and observe that the series  $\sum_{k=0}^n (s_k^{(m)} - \alpha_m) y_j^k$  converge as  $n \rightarrow \infty$  for each  $m \in \mathbb{N}$  and  $j = 1, \dots, \mu$ . Then, by the usual algebra of addition and multiplication of convergent series ([4], Theorem 3.47),

$$\begin{aligned} (1-t) \left| \sum_{k=0}^{\infty} (s_k^{(m)} - \alpha_m) t^k P(t^k) \right| &= (1-t) \left| \sum_{k=0}^{\infty} (s_k^{(m)} - \alpha_m) \sum_{j=1}^{\mu} b_j (t^k)^j \right| \\ &= (1-t) \left| \sum_{k=0}^{\infty} \sum_{j=1}^{\mu} b_j (s_k^{(m)} - \alpha_m) y_j^k \right| \end{aligned}$$

$$\begin{aligned}
&= (1-t) \left| \sum_{j=1}^{\mu} b_j \sum_{k=0}^{\infty} (s_k^{(m)} - \alpha_m) y_j^k \right| \\
&\leq \sum_{j=1}^{\mu} |b_j| (1-t^j) \left| \sum_{k=0}^{\infty} (s_k^{(m)} - \alpha_m) (t^j)^k \right| \\
&= \sum_{j=1}^{\mu} |b_j| |g_m(t^j)|.
\end{aligned}$$

However (9), (10) and the triangle inequality imply that  $\sum_{j=1}^{\mu} |b_j| |g_m(t^j)| < 2\epsilon$  whenever  $T(\epsilon) < t^{\mu} \leq 1$  and  $m > M$ , which proves (6). Hence, by (6) there exist constants  $0 < T_2(\epsilon) < 1$  and  $N_1(\epsilon) \in \mathbb{N}$  such that the inequality (6) will hold simultaneously for both  $p_{\epsilon}$  and  $P_{\epsilon}$  if  $t > T_2(\epsilon)$  and if  $m > N_1(\epsilon)$ . And the inequality (5) together with the inequality (6) (applied to  $p_{\epsilon}$  and  $P_{\epsilon}$ ) yields

$$(11) \quad (1-t) \left| \sum_{k=0}^{\infty} (s_k^{(m)} - \alpha_m) t^k \lambda(t^k) \right| < 4\epsilon$$

if  $\max\{T_1(\epsilon), T_2(\epsilon)\} < t \leq 1$  and if  $m, n > N_1(\epsilon)$ . Finally, in order to complete the proof of Lemma 1 choose the sequence  $t_n = 1 - (n+1)^{-1}$  and observe that

$$(12) \quad (1 - \frac{1}{n+1})^{n+1} < e^{-1} < (1 - \frac{1}{n+1})^n$$

for each  $n \in \mathbb{N}$ . Then there exists a  $N_2(\epsilon) \in \mathbb{N}$  such that  $\max\{T_1(\epsilon), T_2(\epsilon)\} < t_n < 1$  if  $n > N_2(\epsilon)$  and (11), (12) and the definition of  $t\lambda(t)$  applied to the sequence  $(t_n)$  imply that

$$\frac{1}{n+1} \left| \sum_{k=0}^n (s_k^{(m)} - \alpha_m) \right| < 4\epsilon$$

if  $m, n > \max\{N_1(\epsilon), N_2(\epsilon)\}$ . □

The next result will be used several times in the sequel and formulated as a lemma here.

**Lemma 2.** *Let  $(g_n), g_n : [0, 1] \rightarrow \mathbb{R}$  be a sequence of uniformly bounded functions such that each  $g_n$  is non-increasing in  $[0, 1]$  and suppose that there exists a function  $g : [0, 1] \rightarrow \mathbb{R}$  such that for each  $r \in [0, 1]$   $g_n(r) \rightarrow g(r)$  as  $n \rightarrow \infty$ . Then  $\lim_{r \rightarrow 1-} g(r) \geq \lim_{n \rightarrow \infty} g_n(1) = g(1)$ . Furthermore, if  $g : [0, 1] \rightarrow \mathbb{R}$  is continuous then  $(g_n)$  converges uniformly in  $[0, 1]$  to  $g$  as  $n \rightarrow \infty$ .*

*Proof.* The inequality of Lemma 2 is a trivial consequence of properties of monotonic functions. Let  $\epsilon > 0$  be chosen arbitrarily. It is clear that the limit-function  $g : [0, 1] \rightarrow \mathbb{R}$  is non-increasing in  $[0, 1]$ . Since  $g : [0, 1] \rightarrow \mathbb{R}$  is uniformly continuous there exist real numbers  $0 = t_0 < t_1 < \dots < t_m = 1$  such that  $0 \leq g(t_{k-1}) - g(t_k) < \epsilon$  for each  $k = 1, \dots, m$ . Furthermore there exists a  $N(\epsilon) \in \mathbb{N}$  such that for each  $k = 0, 1, \dots, m$   $|g_n(t_k) - g(t_k)| < \epsilon$  whenever  $n > N(\epsilon)$ . Let  $t \in [0, 1]$  be chosen arbitrarily, say  $t_{k-1} < t < t_k$  for some  $1 \leq k \leq m$ . Then, since  $g$  and  $g_n$ ,

$n \in \mathbb{N}$  are non-increasing, the two properties mentioned above yield

$$g(t) - g_n(t) < \epsilon + g(t_{k-1}) - g_n(t) \leq \epsilon + g(t_{k-1}) - g_n(t_{k-1}) < 2\epsilon$$

and

$$g(t) - g_n(t) > g(t_k) - g_n(t) - \epsilon \geq g(t_k) - g_n(t_k) - \epsilon > -2\epsilon$$

if  $n > N(\epsilon)$ . This proves the lemma.  $\square$

In order to introduce the first theorem consider the sequence  $(f_m)$ ,  $f_m(z) = r_m^{-1} k(r_m z)$  where  $k$  denotes the Koebe-function and  $(r_m)$ ,  $m \in \mathbb{N}$  denotes a sequence such that  $0 < r_m < 1$  and  $r_m \rightarrow 1$  as  $m \rightarrow \infty$ . Then each  $f_m$  of the sequence has Hayman-index 0, yet the sequence  $(f_m)$  converges locally uniformly in  $\mathbb{D}$  to the limit-function  $k$  which has Hayman-index 1. Therefore, on the one hand  $|a_n(f_m)|/n \rightarrow 0$  as  $n \rightarrow \infty$  for each  $f_m$  by Hayman's regularity theorem, but on the other hand  $|a_n(f_m)|/n \rightarrow 1$  as  $m \rightarrow \infty$  because the sequence  $(f_m)$  converges locally uniformly in  $\mathbb{D}$  to the Koebe-function. So, if the sequence  $(f_m)$ ,  $f_m \in S$  converges locally uniformly in  $\mathbb{D}$  to a schlicht function  $f$  as  $m \rightarrow \infty$ , the convergence of the Hayman-indexes  $\alpha(f_m)$  of  $f_m \in S$  to the Hayman-index  $\alpha(f)$  of the limit-function  $f \in S$  seems to be an essential hypothesis in order to establish the simultaneous convergence of  $|a_n(f_m)|/n$  to  $\alpha(f_m)$  as  $m, n \rightarrow \infty$ . The first theorem assumes that this minimal hypothesis holds.

**Theorem 1.** *Let  $(f_m)$ ,  $f_m \in S$ ,  $m \in \mathbb{N}$  be given by  $f_m(z) = \sum_{n=1}^{\infty} a_n^{(m)} z^n$  and let  $\epsilon > 0$  be chosen arbitrarily. Furthermore suppose that*

- (i)  *$(f_m)$  converges locally uniformly in  $\mathbb{D}$  to a schlicht function  $f$  as  $m \rightarrow \infty$*
  - (ii)  *$\alpha_m \rightarrow \alpha$  as  $m \rightarrow \infty$  where  $\alpha_m$  and  $\alpha$  denote the Hayman-indexes of  $f_m$  and  $f$  respectively*
- Then there exists a constant  $N(\epsilon)$  only dependent on  $\epsilon$  such that*

$$\left| \frac{a_n^{(m)}}{n} \right| < \epsilon + \sqrt{\alpha_m}$$

*whenever  $m, n > N(\epsilon)$ .*

*Proof.* Let  $\epsilon > 0$  be chosen arbitrarily and consider the case  $\alpha > 0$ , that is the limit function  $f \in S$  is of maximal growth, first. Then, without loss in generality, it may be supposed that for each  $m \in \mathbb{N}$   $f_m \in S$  has Hayman-index  $\alpha_m > 0$  and radius of greatest growth in the direction of the positive real axis. For  $m \in \mathbb{N}$  and  $0 < r < 1$  define  $h_m$  by  $h_m(r) = \log((1-r)^2 r^{-1} f_m(r)) = \sum_{k=1}^{\infty} 2(\gamma_k^{(m)} - k^{-1}) r^k$ , where  $\gamma_k^{(m)}$ ,  $k \in \mathbb{N}$  denote the logarithmic coefficients of  $f_m \in S$ . Then for each  $m \in \mathbb{N}$   $\operatorname{Re}\{h_m(r)\} = \log((1-r)^2 r^{-1} |f_m(r)|)$  is non-increasing in  $[0, 1]$  ([1], chapter 5.5), and by (ii) the limit function defined by  $\lim_{m \rightarrow \infty} \operatorname{Re}\{h_m(r)\} = \log((1-r)^2 r^{-1} |f(r)|)$  for  $r \in [0, 1]$  is continuous in  $[0, 1]$ . Hence, by Lemma 2, the sequence  $(\operatorname{Re} h_m)$  converges uniformly in  $[0, 1]$  as  $m \rightarrow \infty$ . Furthermore by Milin's lemma ([1], chapter 5.4)

$$\sum_{k=1}^n 2 \left( \operatorname{Re} \gamma_k^{(m)} - \frac{1}{k} \right) = \sum_{k=1}^n \left( k \left| \gamma_k^{(m)} \right|^2 - \frac{1}{k} \right) - \sum_{k=1}^n k \left| \gamma_k^{(m)} - \frac{1}{k} \right|^2 \leq \delta < 0.312$$

where  $\delta$  denotes Milin's constant. Therefore Lemma 1 can be applied and asserts that there exists a  $N_1(\epsilon) \in \mathbb{N}$  such that

$$(13) \quad \left| \frac{2}{n+1} \sum_{k=1}^n (n+1-k) \left( \operatorname{Re} \gamma_k^{(m)} - \frac{1}{k} \right) - \log \alpha_m \right| < \epsilon$$

whenever  $m, n > N_1(\epsilon)$ . Since by Basilevich's theorem ([1], Theorem 5.5)

$$(14) \quad \frac{1}{n+1} \sum_{k=1}^n (n+1-k) k \left| \gamma_k^{(m)} - \frac{1}{k} \right|^2 \leq \sum_{k=1}^{\infty} k \left| \gamma_k^{(m)} - \frac{1}{k} \right|^2 \leq -\frac{1}{2} \log \alpha_m$$

for each  $m \in \mathbb{N}$  (13) and (14) imply that

$$(15) \quad \frac{1}{n+1} \sum_{k=1}^n (n+1-k) \left( k \left| \gamma_k^{(m)} \right|^2 - \frac{1}{k} \right) < -\frac{1}{2} \log \alpha_m + \log \alpha_m + \epsilon$$

if  $m, n > N_1(\epsilon)$ . But the second Lebedev-Milin inequality applied to (15) yields

$$\left| a_{n+1}^{(m)} \right| \leq \sum_{k=0}^n \left| b_k^{(m)} \right|^2 \leq (n+1) \exp \left\{ \frac{1}{n+1} \sum_{k=1}^n (n+1-k) \left( k \left| \gamma_k^{(m)} \right|^2 - \frac{1}{k} \right) \right\} < (n+1) e^{\epsilon} \sqrt{\alpha_m}$$

whenever  $m, n > N_1(\epsilon)$  where  $b_k^{(m)}$ ,  $m \in \mathbb{N}$ ,  $k \in \mathbb{N}_0$  denote the coefficients of the functions  $\sqrt{r^{-1} f_m(r)}$ ,  $m \in \mathbb{N}$ . This proves the theorem, if the limit function  $f \in S$  has Hayman-index  $\alpha > 0$ . In order to complete the proof, consider the second case, that is suppose that the limit-function  $f \in S$  has Hayman-index  $\alpha = 0$ . For  $m \in \mathbb{N}$  define  $g_m(r) = r^{-1}(1-r)^2 M_{\infty}(r, f_m)$  and  $g(r) = r^{-1}(1-r)^2 M_{\infty}(r, f)$ . Then each of the functions  $g, g_m : [0, 1] \rightarrow \mathbb{R}$ ,  $m \in \mathbb{N}$  is non-increasing and non-negative ([1], chapter 5.5 Lemma). Moreover they are continuous in  $[0, 1]$  since the functions  $M_{\infty}(\cdot, f)$  and  $M_{\infty}(\cdot, f_m)$ ,  $m \in \mathbb{N}$  clearly are continuous in  $[0, 1]$ . They are also continuous at  $r = 1$  because  $g(r) \rightarrow \alpha$  as  $r \rightarrow 1-$  and  $g_m(r) \rightarrow \alpha_m$  as  $r \rightarrow 1-$  for  $m \in \mathbb{N}$ . In order to show that  $g_m(r) \rightarrow g(r)$  as  $m \rightarrow \infty$  if  $r \in [0, 1]$  observe that this holds for  $r = 1$  and  $r = 0$  because of hypothesis (ii) and since for every  $m \in \mathbb{N}$   $g_m(0) = g(0) = 1$ . So let  $r \in (0, 1)$  and suppose on the contrary that  $g_m(r) \not\rightarrow g(r)$  as  $m \rightarrow \infty$  for some  $r \in (0, 1)$ . Then there exists a subsequence  $(g_{m(k)}(r))$  and a  $\beta \in [0, 1]$  such that

$$(16) \quad \lim_{k \rightarrow \infty} g_{m(k)}(r) = \beta \neq g(r).$$

However the subsequence can be chosen in such a way that  $M_{\infty}(r, f_{m(k)}) = |f_{m(k)}(z_{m(k)})|$  where  $z_{m(k)} \rightarrow z$  as  $k \rightarrow \infty$  for some  $z$ ,  $z_{m(k)}$  with  $|z| \leq r$ ,  $|z_{m(k)}| \leq r$ ,  $k \in \mathbb{N}$ , and such that (16) holds. But, because of the uniform convergence of the sequence  $(f_m)$  in  $|z| \leq r$ , there exists a  $N_1(\epsilon) \in \mathbb{N}$  such that

$$(17) \quad \left| |f_{m(k)}(z_{m(k)})| - |f(z)| \right| = \left| M_{\infty}(r, f_{m(k)}) - |f(z)| \right| < \frac{\epsilon}{3}.$$

if  $k > N_1(\epsilon)$ . Let  $w$ ,  $|w| \leq r$  be a point such that  $|f(w)| = M_{\infty}(r, f)$ . Then, since  $(f_m)$  converges uniformly in  $|z| \leq r$  as  $m \rightarrow \infty$ , there exists a  $N_2(\epsilon) \in \mathbb{N}$  such that

$$(18) \quad \left| |f_{m(k)}(w)| - M_{\infty}(r, f) \right| < \frac{\epsilon}{3}$$

if  $k > N_2(\epsilon)$ . Therefore (17), (18) and the fact, that  $f$  and  $f_{m(k)}$ ,  $k \in \mathbb{N}$  take on their maxima if  $|z| \leq r$  at  $z = w$  and  $z = z_{m(k)}$ ,  $k \in \mathbb{N}$  respectively, yield

$$(19) \quad 0 \leq M_\infty(r, f_{m(k)}) - |f_{m(k)}(w)| \leq M_\infty(r, f_{m(k)}) - |f_{m(k)}(w)| + M_\infty(r, f) - |f(z)| < \frac{2\epsilon}{3}$$

whenever  $k > \max\{N_1(\epsilon), N_2(\epsilon)\}$ . Now (18) and (19) imply that

$$|M_\infty(r, f_{m(k)}) - M_\infty(r, f)| \leq |M_\infty(r, f_{m(k)}) - |f_{m(k)}(w)|| + ||f_{m(k)}(w)| - M_\infty(r, f)| < \epsilon$$

if  $k > \max\{N_1(\epsilon), N_2(\epsilon)\}$  which proves that  $g_{m(k)}(r) \rightarrow g(r)$  as  $k \rightarrow \infty$ , a contradiction to (16). So  $g_m(r) \rightarrow g(r)$  as  $m \rightarrow \infty$  for  $r \in [0, 1]$  and an appeal to Lemma 2 shows that  $g_m \rightarrow g$  uniformly on  $[0, 1]$  as  $m \rightarrow \infty$ . But by Prawitz' theorem([1], Theorem 2.22)

$$\frac{d}{dr} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f_m(re^{it})| dt \right\} \leq \frac{1}{r} M_\infty(r, f_m) = g_m(r) \frac{1}{(1-r)^2}$$

if  $m \in \mathbb{N}$ . Integrating the last inequality from  $r_0$  to  $r$  where  $0 < r_0 < r < 1$  yields

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f_m(re^{it})| dt - \frac{1}{2\pi} \int_0^{2\pi} |f_m(r_0 e^{it})| dt &\leq g_m(r_0) \int_{r_0}^r (1-t)^{-2} dt \\ &= g_m(r_0) \left( \frac{1}{1-r} - \frac{1}{1-r_0} \right). \end{aligned}$$

It follows that

$$(20) \quad (1-r) \frac{1}{2\pi} \int_0^{2\pi} |f_m(re^{it})| dt \leq (1-r) \frac{1}{2\pi} \int_0^{2\pi} |f_m(r_0 e^{it})| dt + g_m(r_0)$$

for  $m \in \mathbb{N}$ . Because  $g_m \rightarrow g$  uniformly on  $[0, 1]$  as  $m \rightarrow \infty$ , because each  $g_m$  is continuous on  $[0, 1]$  and because  $g(1) = 0$  there exists a  $N_3(\epsilon) \in \mathbb{N}$  such that

$$(21) \quad 0 < g_m(r) < \epsilon$$

whenever  $m > N_3(\epsilon)$  and  $1 - N_3(\epsilon)^{-1} \leq r < 1$ . Therefore, if  $r_0$  in (20) is chosen as  $r_0 = 1 - N_3(\epsilon)^{-1}$  then, since  $f_m \rightarrow f$  locally uniformly in  $\mathbb{D}$  as  $m \rightarrow \infty$ , there exists a  $N_4(\epsilon) \in \mathbb{N}$ ,  $N_4(\epsilon) > N_3(\epsilon)$  such that for  $t \in [0, 2\pi]$   $|f_m(r_0 \exp(it)) - f(r_0 \exp(it))| < \epsilon$  if  $m > N_4(\epsilon)$ . Then there exists also a  $N_5(\epsilon) \in \mathbb{N}$ ,  $N_5(\epsilon) > N_4(\epsilon)$  such that

$$\begin{aligned} (1-r) \frac{1}{2\pi} \int_0^{2\pi} |f_m(r_0 e^{it})| dt &< (1-r) \left\{ \epsilon + \frac{1}{2\pi} \int_0^{2\pi} |f(r_0 e^{it})| dt \right\} \\ &< \epsilon \end{aligned}$$

whenever  $1 - N_5(\epsilon)^{-1} \leq r < 1$  and  $m > N_5(\epsilon)$ . In order to complete the proof of Theorem 1 let  $r = r_n = 1 - n^{-1}$  and observe that by (20), (21) and the last inequality the Cauchy inequality for the coefficients yields

$$\left| \frac{a_n^{(m)}}{n} \right| = |a_n^{(m)}| (1 - r_n) \leq r_n^{-n} (1 - r_n) \frac{1}{2\pi} \int_0^{2\pi} |f_m(r_n e^{it})| dt$$

$$\leq r_n^{-n} \left( (1-r_n) \frac{1}{2\pi} \int_0^{2\pi} |f_m(r_0 e^{it})| dt + g_m(r_0) \right) < 4(\epsilon + \epsilon)$$

if  $m, n > \max\{N_1(\epsilon), N_2(\epsilon), N_5(\epsilon)\}$ . This completes the proof of Theorem 1 if  $\alpha = 0$ .  $\square$

There are some interesting applications of Theorem 1 to asymptotic extremal problems concerning the class of schlicht functions.

**Example 1.** (Asymptotic Bieberbach Conjecture) If for each  $n \in \mathbb{N}$   $f_n$  is assumed to be a schlicht function that maximizes the modulus of the  $n$ -th coefficient then Theorem 1 reveals that in order to prove the asymptotic Bieberbach conjecture (now superseded by de Branges theorem) it only has to be shown that for any subsequence  $(f_{n(k)})$  of  $(f_n)$  that converges locally uniformly in  $\mathbb{D}$  to some schlicht function  $f$  the Hayman-indexes  $\alpha(f_{n(k)})$  of  $f_{n(k)} \in S$  converge to the Hayman-index  $\alpha(f)$  of  $f \in S$  as  $k \rightarrow \infty$  (for more details about the asymptotic Bieberbach conjecture see [1], chapter 2.12).

Here is another example how Theorem 1 can be applied to asymptotic extremal problems for schlicht functions.

**Example 2.** (Asymptotic Zalcman Conjecture) Define functionals  $F_n$ ,  $n \in \mathbb{N}$  for  $f \in S$  by  $F_n(f) = |a_n(f)^2 - a_{2n-1}(f)|$ . Then Zalcman's conjecture asserts that  $F_n(f) \leq (n-1)^2$  for  $f \in S$ . Suppose that  $f_n$ ,  $n \in \mathbb{N}$  maximizes  $F_n$  within the class of schlicht functions and that there were a subsequence  $(f_{n(k)})$  that converges to a schlicht function  $f$  of slow growth. Then the sequence  $(f_{n(k)})$  satisfies hypothesis (ii) of Theorem 1 (by Lemma 2) and consequently  $(n(k)-1)^{-2} F_{n(k)}(f_{n(k)}) \rightarrow 0$  as  $k \rightarrow \infty$  by Theorem 1, a contradiction since the Koebe-function  $k$  satisfies  $F_n(k) = (n-1)^2 > 0$  if  $n \geq 2$ . This argument shows that the functionals  $(F_n)$  (and certain other sequences of coefficient functionals), or rather their extremal functions, have no accumulation points of slow growth (with respect to the topology of uniform convergence on compact subsets of  $\mathbb{D}$ ).

The next lemma will be used in the proof of Theorem 2, however it is of interest in itself, since it provides an extension of an earlier result of Bazilevich ([5]) concerning the case of equality in his theorem.

**Lemma 3.** *Let  $f \in S$  have Hayman-index  $\alpha > 0$  and suppose that  $g : \Delta \rightarrow \mathbb{C}$  defined by  $g(z) = f(z^{-1})^{-1}$  is a full mapping, that is  $g \in \tilde{\Sigma}$ . Furthermore let  $\gamma_n$ ,  $n \in \mathbb{N}$  denote the logarithmic coefficients of  $f \in S$  and let  $\exp(i\theta)$ ,  $\theta \in [0, 2\pi)$  denote the direction of greatest growth of  $f \in S$ . Then*

$$\sum_{n=1}^{\infty} n \left| \gamma_n - \frac{1}{n} \exp(-in\theta) \right|^2 = -\frac{1}{2} \log \alpha$$

*Proof.* In order to prove the lemma it obviously suffices to consider the case that  $f \in S$  is not a rotation of the Koebe-function. Let  $A_n(z) = \sum_{k=1}^{\infty} \gamma_{nk} z^k$ ,  $z \in \mathbb{D}$  where  $\gamma_{nk}$ ,  $k, n \in \mathbb{N}$  denote



the Grunsky coefficients of  $g \in \tilde{\Sigma}$ . For  $n \in \mathbb{N}$  and an arbitrarily chosen  $\mathbf{u} = (u_1, u_2, \dots) \in \ell^2$  define a sequence of continuous linear mappings  $T_n : \ell^2 \rightarrow \ell^2$   $T_n \mathbf{u} = \mathbf{v} = (v_1, v_2, \dots)$  by  $v_k = \sqrt{k} \sum_{j=1}^n \sqrt{j} \gamma_{kj} u_j$ ,  $k \in \mathbb{N}$ . Further, let  $\|\cdot\|$  and  $(\cdot, \cdot)$  denote the norm and inner product of the Hilbert space  $\ell^2$ . The mappings  $T_n$ ,  $n \in \mathbb{N}$  are well defined because the strong Grunsky inequality ([1], chapter 4.3 formula (10)) yields

$$(22) \quad \|T_n \mathbf{u}\|^2 = (T_n \mathbf{u}, T_n \mathbf{u}) = \sum_{k=1}^{\infty} |v_k|^2 = \sum_{k=1}^{\infty} k \left| \sum_{j=1}^n \gamma_{kj} \sqrt{j} u_j \right|^2 \leq \sum_{j=1}^n |u_j|^2 \leq \|\mathbf{u}\|^2$$

that is  $T_n \mathbf{u} \in \ell^2$  and  $\|T_n\| \leq 1$  for  $n \in \mathbb{N}$ . Now define a dense subset  $U \subseteq \ell^2$  by  $U = \bigcup_{k \in \mathbb{N}} U_k$  where  $U_k = \{\mathbf{u} = (u_1, u_2, \dots) \in \ell^2 : u_j = 0 \text{ for } j > k\}$ ,  $k \in \mathbb{N}$  and observe that the sequences  $(T_n \mathbf{u})$  converge pointwise for each  $\mathbf{u} \in U$  as  $n \rightarrow \infty$ . Consequently the Banach-Steinhaus theorem ([6], Theorem 2.7) implies that the mapping  $T : \ell^2 \rightarrow \ell^2$   $T \mathbf{u} = \lim_{n \rightarrow \infty} T_n \mathbf{u}$  is well-defined for each  $\mathbf{u} \in \ell^2$  and is linear and continuous. Since the vectors  $\mathbf{e}_k = (\delta_{1,k}, \delta_{2,k}, \dots) \in \ell^2$ ,  $k \in \mathbb{N}$ , where  $\delta_{i,k} = 1$  if  $i = k$  and  $\delta_{i,k} = 0$  if  $i \neq k$  define an orthonormal basis of  $\ell^2$  and since pointwise convergence implies weak convergence Parseval's theorem yields

$$(23) \quad \begin{aligned} \infty > \|T \mathbf{u}\|^2 &= \sum_{k=1}^{\infty} |(T \mathbf{u}, \mathbf{e}_k)|^2 \\ &= \sum_{k=1}^{\infty} \left| \lim_{n \rightarrow \infty} (T_n \mathbf{u}, \mathbf{e}_k) \right|^2 \\ &= \sum_{k=1}^{\infty} \left| \lim_{n \rightarrow \infty} \sqrt{k} \sum_{j=1}^n \gamma_{kj} \sqrt{j} u_j \right|^2 \\ &= \sum_{k=1}^{\infty} k \left| \sum_{j=1}^{\infty} \gamma_{kj} \sqrt{j} u_j \right|^2 \end{aligned}$$

for each  $\mathbf{u} \in \ell^2$ . On the other hand, since  $g \in \tilde{\Sigma}$ , equality holds in the strong Grunsky inequalities and hence equality holds in (22), which means that

$$(24) \quad \infty > \|T \mathbf{u}\|^2 = \lim_{n \rightarrow \infty} \|T_n \mathbf{u}\|^2 = \lim_{n \rightarrow \infty} \sum_{j=1}^n |u_j|^2 = \|\mathbf{u}\|^2.$$

Therefore, if  $u_j = z(\sqrt{j})^{-1}$ ,  $j \in \mathbb{N}$ , by the definition of  $A_n(z)$  and by (23) and (24)

$$(25) \quad \sum_{n=1}^{\infty} n |A_n(z)|^2 = \sum_{k=1}^{\infty} k \left| \sum_{j=1}^{\infty} \gamma_{kj} z^j \right|^2 = \sum_{j=1}^{\infty} \frac{1}{j} |z_j|^2 = -\log(1 - |z|^2)$$

if  $g \in \tilde{\Sigma}$ . Now let  $w(r) = r \exp(i\theta)$ ,  $0 < r < 1$ , then  $|w(r)| = r$ . By (25) and since  $\sum_{n=1}^{\infty} n A_n(z) z^n = 2 \log(\bar{z}^{-1} f(z)) - \log f'(z)$  for  $z \in \mathbb{D}$  (see [1], proof of Theorem 5.5) it follows that

$$\begin{aligned}
 \sum_{n=1}^{\infty} n \left| A_n(w(r)) - \frac{1}{n} \overline{w(r)}^n \right|^2 &= \sum_{n=1}^{\infty} n |A_n(w(r))|^2 - 2 \operatorname{Re} \left\{ \sum_{n=1}^{\infty} A_n(w(r)) w(r)^n \right\} + \\
 &\quad \sum_{n=1}^{\infty} \frac{1}{n} |w(r)|^{2n} \\
 &= -2 \log(1 - |w(r)|^2) + 2 \log \frac{|w(r)^2 f'(w(r))|}{|f(w(r))^2|} \\
 (26) \quad &= 2 \log \left( \frac{|f'(w(r))|}{1+r} (1-r)^3 \right) - 4 \log \left( \frac{|f(w(r))|}{r} (1-r)^2 \right).
 \end{aligned}$$

In order to complete the proof it has to be shown that

$$(27) \quad \lim_{r \rightarrow 1-} \frac{|f'(w(r))|}{1+r} (1-r)^3 = \alpha$$

if  $\alpha > 0$ . On the one hand, if  $0 < r < 1$ , by [1], Theorem 2.7

$$(28) \quad \frac{|f'(w(r))| (1-r)^3}{1+r} \leq \frac{|f(w(r))| (1-r)^2}{r} \leq 1,$$

and since  $\alpha < 1$  by [1], chapter 2.3, p. 33

$$(29) \quad \frac{\partial}{\partial r} \log \left( \frac{(1-r)^3}{1+r} |f'(w(r))| \right) = \frac{\partial}{\partial r} \log |f'(w(r))| - \frac{4+2r}{1-r^2} < 0.$$

Hence by (29)  $|f'(w(r))| (1-r)^3 (1+r)^{-1}$  is strictly decreasing, by (28) it is bounded and consequently  $|f'(w(r))| (1-r)^3 (1+r)^{-1} \rightarrow \vartheta$  as  $r \rightarrow 1-$  for some  $0 \leq \vartheta \leq 1$ . But since  $|f(w(r))| r^{-1} (1-r)^2 \rightarrow \alpha$  as  $r \rightarrow 1-$  (28) implies that

$$(30) \quad 0 \leq \vartheta \leq \alpha.$$

On the other hand by the fundamental theorem of calculus and by de L'Hospital's rule ([4], Theorem 5.13)

$$\begin{aligned}
 \alpha = \lim_{r \rightarrow 1-} (1-r)^2 |f(w(r))| &\leq \lim_{r \rightarrow 1-} (1-r)^2 \int_0^r |f'(t \exp(i\theta))| dt \\
 &= \lim_{r \rightarrow 1-} \frac{\frac{d}{dr} \int_0^r |f'(t \exp(i\theta))| dt}{\frac{d}{dr} \{(1-r)^{-2}\}} \\
 &= \lim_{r \rightarrow 1-} \frac{1}{2} (1-r)^3 |f'(w(r))| = \vartheta.
 \end{aligned}$$

However then, by (30),  $\alpha = \vartheta$ , which proves (27). To complete the proof of the lemma observe that

$$(31) \quad \lim_{r \rightarrow 1-} A_n(r \exp(i\theta)) = 2\gamma_n - \frac{1}{n} \exp(-in\theta)$$

(see [1], proof of Theorem 5.5). Now let  $r \rightarrow 1-$  on both sides of equation (26) then by (27), (31) and by definition of the Hayman-index

$$\begin{aligned} 4 \sum_{n=1}^{\infty} n \left| \gamma_n - \exp(-in\theta) \frac{1}{n} \right|^2 &= \lim_{r \rightarrow 1-} \left\{ 2 \log \left( \frac{|f'(w(r))|}{1+r} (1-r)^3 \right) - 4 \log \left( \frac{|f'(w(r))|}{r} (1-r)^2 \right) \right\} \\ &= 2 \log \alpha - 4 \log \alpha. \end{aligned}$$

This proves the lemma.  $\square$

Theorem 1 already provides a uniform convergence result for the coefficients if the limit function is of slow growth. With the extended version of Bazilevich's theorem at hand this result can be extended to the case that the limit function is of maximal growth.

**Theorem 2.** Let  $(f_m), f_m \in S$  be given by  $f_m(z) = \sum_{n=1}^{\infty} a_n^{(m)} z^n$  and suppose that

(i)  $(f_m)$  converges locally uniformly in  $\mathbb{D}$  to a schlicht function  $f$  of maximal growth as  $m \rightarrow \infty$  such that the function  $z \rightarrow f(z^{-1})^{-1}$ ,  $z \in \Delta$  is a full mapping

(ii)  $\alpha_m \rightarrow \alpha$  as  $m \rightarrow \infty$  where  $\alpha_m$  and  $\alpha$  denote the Hayman-indexes of  $f_m$  and  $f$  respectively

Then for every  $\epsilon > 0$  there exists a constant  $N(\epsilon)$  only dependent on  $\epsilon$  such that

$$\left| \frac{|a_n^{(m)}|}{n} - \alpha_m \right| < \epsilon$$

whenever  $m, n > N(\epsilon)$ .

*Proof.* Let  $0 < \epsilon < \alpha$  be given arbitrarily and suppose that the limit function  $f \in S$  is given by  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ ,  $z \in \mathbb{D}$ . Since the limit function  $f \in S$  was supposed to be of maximal growth type, by (ii) there exists a constant  $N_1(\epsilon) \in \mathbb{N}$  such that

$$(32) \quad 0 < \alpha - \epsilon < \alpha_m < \alpha + \epsilon$$

whenever  $m > N_1(\epsilon)$ . Hence, without loss in generality, it may be supposed, that each  $f_m$ ,  $m \in \mathbb{N}$  has Hayman-index  $\alpha_m > 0$  and radius of greatest growth in the direction of the positive real axis. For each  $m \in \mathbb{N}$  define  $h_m : \mathbb{D} \rightarrow \mathbb{C}$  by  $h_m(z) = \sum_{n=1}^{\infty} n |\gamma_n^{(m)} - n^{-1}|^2 z^n$  and  $g, g_m : \mathbb{D} \rightarrow \mathbb{C}$  by  $g_m(z) = \log\{z^{-1} f_m(z)\} = 2 \sum_{n=1}^{\infty} \gamma_n^{(m)} z^n$  and  $g(z) = \log\{z^{-1} f(z)\} = 2 \sum_{n=1}^{\infty} \gamma_n z^n$  respectively. Here  $\gamma_n^{(m)}$ ,  $n \in \mathbb{N}$  and  $\gamma_n$ ,  $n \in \mathbb{N}$  denote the logarithmic coefficients of  $f_m \in S$  and  $f \in S$  respectively. The strategy of proof is to first apply Lemma 1 to the sequence  $(h_m)$ . In order to do that observe that by (32) and by Bazilevich's theorem ([1], Theorem 5.5) the family  $\{h_m\}$  is uniformly bounded in  $\overline{\mathbb{D}}$  and therefore there exists a subsequence  $(h_{m(k)})$  and an analytic function  $h : \mathbb{D} \rightarrow \mathbb{C}$  such that  $h_{m(k)} \rightarrow h$  locally uniformly in  $\mathbb{D}$  as  $k \rightarrow \infty$ . Let  $h(z) = \sum_{k=1}^{\infty} \beta_k z^k$  and consider an arbitrarily chosen coefficient  $\beta_n$ ,  $n \in \mathbb{N}$ . Then  $n |\gamma_n^{(m(k))} - n^{-1}|^2 \rightarrow \beta_n$  as  $k \rightarrow \infty$  because  $h_{m(k)} \rightarrow h$  locally uniformly in  $\mathbb{D}$  as  $k \rightarrow \infty$ . It is well-known that the coefficients  $\gamma_n^{(m)}$ ,  $m \in \mathbb{N}$  satisfy the equations  $a_n^{(m)} = n \gamma_n^{(m)} + \sum_{k=1}^{n-1} k \gamma_k^{(m)} a_{n-k+1}^{(m)}$  for  $m \in \mathbb{N}$ . By hypothesis

(i)  $a_n^{(m)} \rightarrow a_n$  as  $m \rightarrow \infty$  for each  $n \in \mathbb{N}$  and inductively (by  $n$ ) it follows that

$$\lim_{m \rightarrow \infty} \gamma_n^{(m)} = \lim_{m \rightarrow \infty} \frac{1}{n} \left( a_n^{(m)} - \sum_{k=1}^{n-1} k \gamma_k^{(m)} a_{n-k+1}^{(m)} \right) = \frac{1}{n} \left( a_n - \sum_{k=1}^{n-1} k \gamma_k a_{n-k+1} \right) = \gamma_n.$$

Consequently also  $n|\gamma_n^{(m(k))} - n^{-1}|^2 \rightarrow n|\gamma_n - n^{-1}|^2$  as  $k \rightarrow \infty$  and by the identity theorem  $\beta_n = n|\gamma_n - n^{-1}|^2$  for each  $n \in \mathbb{N}$ . Because the subsequence  $(h_{m(k)})$  was chosen arbitrarily each subsequence  $(h_{m(k)})$  of the sequence  $(h_m)$  will converge locally uniformly in  $\mathbb{D}$  to  $h$  as  $k \rightarrow \infty$  and by a result of Montel ([7], Theorem 2.4.2) the whole sequence  $(h_m)$  converges to the limit-function  $h$  locally uniformly in  $\mathbb{D}$  as  $m \rightarrow \infty$ . Clearly  $-h$  and  $-h_m$ ,  $m \in \mathbb{N}$  are non-increasing in  $[0, 1]$  and by Bazilevich's theorem and Abel's limit theorem  $-h$  and  $-h_m$ ,  $m \in \mathbb{N}$  are continuous in  $[0, 1]$ . In order to show that  $h_m(1) \rightarrow h(1)$  as  $m \rightarrow \infty$  consider an arbitrarily chosen accumulation point of the sequence  $(h_m(1))$ , that is consider an arbitrarily chosen subsequence  $(h_{m(k)}(1))$  such that  $h_{m(k)}(1) \rightarrow \lambda$  as  $k \rightarrow \infty$  for some  $\lambda \geq 0$ . Then on the one hand by Lemma 2 and hypothesis (ii)

$$(33) \quad \lim_{r \rightarrow 1-} \{-h(r)\} \geq \lim_{k \rightarrow \infty} \{-h_{m(k)}(1)\} \geq \lim_{k \rightarrow \infty} \frac{1}{2} \log \alpha_{m(k)} = \frac{1}{2} \log \alpha$$

and on the other hand  $h(1) = -(1/2) \log \alpha$  by Lemma 3 and Abel's limit theorem. Consequently equality holds in (33) for each convergent subsequence  $(h_{m(k)}(1))$  of the sequence  $(h_m(1))$  and therefore  $h_m(1) \rightarrow h(1)$  as  $m \rightarrow \infty$ . Now Lemma 2 can be applied and implies that  $h_m \rightarrow h$  uniformly in  $[0, 1]$  as  $m \rightarrow \infty$  and by Lemma 1 applied to the functions  $h_m$  there exists a  $N_2(\epsilon) \in \mathbb{N}$  such that

$$(34) \quad 0 \leq -\frac{1}{2} \log \alpha_m - \sum_{k=1}^n k \left| \gamma_k^{(m)} - \frac{1}{k} \right|^2 \leq -\frac{1}{2} \log \alpha_m - \frac{1}{n+1} \sum_{k=1}^n (n+1-k) k \left| \gamma_k^{(m)} - \frac{1}{k} \right|^2 < \epsilon$$

if  $m, n > N_2(\epsilon)$ . By subtracting the two inequalities of (34) one obtains

$$(35) \quad 0 \leq \frac{1}{n+1} \sum_{k=1}^n k^2 \left| \gamma_k^{(m)} - \frac{1}{k} \right|^2 < 2\epsilon$$

whenever  $m, n > N_1(\epsilon)$ . So, for  $m \in \mathbb{N}$ , consider the functions  $g_m, F_m$  defined by  $g_m(r) + 2 \log(1-r) = \sum_{n=1}^{\infty} \lambda_n^{(m)} r^n$  where  $\lambda_k^{(m)} = 2(\gamma_k^{(m)} - n^{-1})$  and  $F_m(r) = \exp(g_m(r) + 2 \log(1-r)) = r^{-1}(1-r)^2 f_m(r) = \sum_{n=0}^{\infty} b_n^{(m)} r^n$ . Then  $n b_n^{(m)} = \sum_{k=1}^n \lambda_k^{(m)} b_{n-k}^{(m)}$  for  $m \in \mathbb{N}$ , and inductively it follows that

$$(36) \quad \frac{1}{n+1} \sum_{k=1}^n k b_k^{(m)} = \frac{1}{n+1} \sum_{j=1}^n j \lambda_j^{(m)} s_{n-j}^{(m)}$$

where

$$(37) \quad s_k^{(m)} = \sum_{j=0}^k b_j^{(m)} = a_{k+1}^{(m)} - a_k^{(m)} \quad m \in \mathbb{N}.$$

The Cauchy-Schwarz inequality applied to (36) yields

$$(38) \quad \left| \frac{1}{n+1} \sum_{k=1}^n k b_k^{(m)} \right|^2 \leq \left( \frac{4}{n+1} \sum_{k=1}^n k^2 \left| \gamma_k^{(m)} - \frac{1}{k} \right|^2 \right) \left( \frac{1}{n+1} \sum_{k=0}^{n-1} |s_k^{(m)}|^2 \right).$$

But (32) and [1], Theorem 5.10 imply that  $|s_n^{(m)}|^2 \leq (\alpha - \epsilon)^{-1} \exp(2\delta)$  if  $m > N_1(\epsilon)$ , where  $\delta$  denotes Milin's constant. Hence, if  $\delta_n^{(m)}$ ,  $m, n \in \mathbb{N}$  is defined by  $\delta_n^{(m)} = (n+1)^{-1} \sum_{k=1}^n k b_k^{(m)}$ , (35) and (38) imply that

$$(39) \quad |\delta_n^{(m)}| = \left| \frac{1}{n+1} \sum_{k=1}^n k b_k^{(m)} \right| < \sqrt{8\epsilon(\alpha - \epsilon)^{-1} \exp(2\delta)}$$

whenever  $m, n > M(\epsilon)$  where  $M(\epsilon) = \max\{N_1(\epsilon), N_2(\epsilon)\}$ . From now on the lines of proof essentially follow Tauber's well-known proof of his second theorem, however adapted to sequences of functions. Tauber([8], p.276 or [9], chapter 7) obtains the two formulas

$$\sum_{k=1}^n b_k^{(m)} = \sum_{k=1}^n \left(1 + \frac{1}{k}\right) \delta_k^{(m)} - \sum_{k=1}^n \delta_{k-1}^{(m)} = \left(1 + \frac{1}{n}\right) \delta_n^{(m)} + \sum_{k=1}^{n-1} \frac{1}{k} \delta_k^{(m)}$$

and

$$F_m(r) - b_0^{(m)} = \sum_{n=1}^{\infty} b_n^{(m)} r^n = \sum_{k=1}^{\infty} \frac{\delta_k^{(m)}}{k} r^k + (1-r) \sum_{k=1}^{\infty} \delta_k^{(m)} r^k$$

which hold if  $n \geq 2$  and  $r \in (0, 1)$ . The first formula easily can be verified inductively and the second follows by a straightforward calculation. By subtracting the last two equations it follows that

$$(40) \quad F_m(r) - s_n^{(m)} = (1-r) \sum_{k=1}^{\infty} \delta_k^{(m)} r^k + \sum_{k=n}^{\infty} \frac{\delta_k^{(m)}}{k} r^k + \sum_{k=1}^{n-1} \frac{\delta_k^{(m)}}{k} (r^k - 1) - \left(1 + \frac{1}{n}\right) \delta_n^{(m)}$$

which holds if  $m \in \mathbb{N}$ ,  $n \geq 2$  and if  $r \in (0, 1)$ . The Cesaro-means  $\sigma_n^{(m)}$ ,  $m \in \mathbb{N}$ ,  $n \in \mathbb{N}_0$  are defined by  $\sigma_n^{(m)} = (n+1)^{-1} \sum_{k=0}^n s_k^{(m)}$ , hence  $\delta_n^{(m)} = s_n^{(m)} - \sigma_n^{(m)}$  and therefore (40) can be written in the form

$$(41) \quad F_m(r) - \sigma_n^{(m)} = (1-r) \sum_{k=1}^{\infty} \delta_k^{(m)} r^k + \sum_{k=n}^{\infty} \frac{\delta_k^{(m)}}{k} r^k + \sum_{k=1}^{n-1} \frac{\delta_k^{(m)}}{k} (r^k - 1) - \frac{\delta_n^{(m)}}{n}$$

whenever  $n \geq 2$  and  $r \in (0, 1)$ . In order to simplify notation let  $\varepsilon = \sqrt{8\epsilon(\alpha - \epsilon)^{-1} \exp(2\delta)}$  and observe that by (39), there exists a constant  $L > 0$  such that  $|\delta_n^{(m)}| < L$  if  $m, n \in \mathbb{N}$ . Then, since  $|r^k - 1| \leq k(1-r)$  for  $k \in \mathbb{N}$ , (39) and (41) yield

$$(42) \quad \begin{aligned} |F_m(r) - \sigma_n^{(m)}| &\leq 2(1-r) \sum_{k=1}^{M(\epsilon)} |\delta_k^{(m)}| r^k + 2(1-r) \sum_{k=M(\epsilon)+1}^{\infty} |\delta_k^{(m)}| r^k + \sum_{k=n}^{\infty} \frac{|\delta_k^{(m)}|}{k} r^k + \frac{|\delta_n^{(m)}|}{n} \\ &< 2(1-r)M(\epsilon)L + 2\varepsilon + \frac{1}{1-r} \frac{\varepsilon}{n} + \frac{\varepsilon}{n} \end{aligned}$$

if  $m, n > M(\epsilon)$  and  $r \in (0, 1)$ . Now let  $F : [0, 1] \rightarrow \mathbb{R}$  be defined by  $F(r) = (1-r)^2 r^{-1} f(r)$  and remember that  $F_m(r) = (1-r)^2 r^{-1} f_m(r)$  for  $m \in \mathbb{N}$ . Then hypothesis (ii) and Lemma 2 imply that the sequence  $(|F_m|)$  converges uniformly in  $[0, 1]$  to  $|F|$  as  $m \rightarrow \infty$ , that is there exists a  $N_3(\epsilon) \in \mathbb{N}$  such that  $\|F_m(r) - |F(r)|\| < \epsilon$  whenever  $m > N_3(\epsilon)$ . But since  $|F|$  is continuous in  $[0, 1]$  there exists a  $\rho > 0$  such that  $\|F(r) - \alpha\| < \epsilon$  if  $1-r < \rho$ . By (32) it follows that  $|\alpha - \alpha_m| < \epsilon$  if  $m > N_1(\epsilon)$  and hence

$$(43) \quad \|F_m(r) - \alpha_m\| \leq \|F_m(r) - |F(r)|\| + \|F(r) - \alpha\| + |\alpha - \alpha_m| < 3\epsilon$$

whenever  $m > \max\{N_1(\epsilon), N_3(\epsilon)\}$  and  $1-r < \rho$ . Finally, in order to complete the proof, let  $r = 1 - n^{-1}$  and observe that there exists a  $N_4(\epsilon) \in \mathbb{N}$  such that  $1-r = n^{-1} < \rho$  if  $n > N_4(\epsilon)$ . Similarly, if  $1-r = n^{-1}$ , there exists a  $N_5(\epsilon) \in \mathbb{N}$  such that

$$(44) \quad (1-r)M(\epsilon)L = \frac{M(\epsilon)L}{n} < \epsilon$$

if  $n > N_5(\epsilon)$ . Observe that by (37)  $\sigma_n^{(m)} = (n+1)^{-1} \sum_{k=0}^n s_k^{(m)} = (n+1)^{-1} a_{n+1}^{(m)}$  for  $m, n \in \mathbb{N}$  and that therefore

$$\left| \alpha_m - \frac{a_{n+1}^{(m)}}{n+1} \right| = \left| \alpha_m - |\sigma_n^{(m)}| \right| \leq \left| F_m(1 - \frac{1}{n}) - |\sigma_n^{(m)}| \right| + \left| \alpha_m - F_m(1 - \frac{1}{n}) \right| < (2\epsilon + 4\epsilon) + 3\epsilon$$

by (42), (43) and (44) whenever  $m, n > \max\{N_1(\epsilon), N_2(\epsilon), N_3(\epsilon), N_4(\epsilon), N_5(\epsilon)\}$ . This completes the proof.  $\square$

Several questions arise, the most interesting perhaps whether Theorem 2 remains true if the function  $g : \Delta \rightarrow \mathbb{C}$  defined by  $g(z) = f(z^{-1})^{-1}$  is not a full mapping (where  $f \in S$  denotes the limit function of Theorem 2). The proof of Theorem 2 suggests that this might not be true since for this class of functions strict inequality holds in Basilevich's theorem. The observations made so far suggest the following definition of an approximation measure for schlicht functions.

**Definition 1.** A schlicht function  $f \in S$  will be called not badly approximable if for any sequence  $(f_n)$ ,  $f_n \in S$  such that  $f_n \rightarrow f$  locally uniformly in  $\mathbb{D}$  and  $\alpha(f_n) \rightarrow \alpha(f)$  as  $n \rightarrow \infty$  and for any  $\epsilon > 0$  there exists a number  $N \in \mathbb{N}$  (dependent only on  $\epsilon$  and the sequence  $(f_n)$ ) so that  $\left| k^{-1} a_k^{(n)} - \alpha(f) \right| < \epsilon$  whenever  $k, n > N$ . Here it is assumed that  $f_n(z) = \sum_{k=1}^{\infty} a_k^{(n)} z^k$  for  $z \in \mathbb{D}$  and  $n \in \mathbb{N}$ .

In the terminology of Definition 1 every schlicht function  $f \in S$  of slow growth type is not badly approximable by Theorem 1. By Theorem 2 every schlicht function  $f \in S$  whose associated inverted function  $g : \Delta \rightarrow \mathbb{C}$  defined by  $g(z) = f(z^{-1})^{-1}$  is a full mapping is not badly approximable either. The full-mapping property and boundedness of the image regions for instance are geometric properties of the image regions of schlicht functions and so Theorems 1 and 2 also allow for a geometric interpretation.

The use of approximation measures has a long-standing tradition in the theory of diophantine approximation and the current paper was actually inspired by that. To mention just one of the approximation measures of diophantine approximation, an irrational number  $\lambda$  is called badly approximable if and only if there is a constant  $c = c(\lambda) > 0$  such that  $|\lambda - p/q| > c/q^2$  for every rational number  $p/q$  (see [10], chapter I.5). There are continuum many badly approximable irrationals and continuum many not badly approximable irrationals and this particular approximation measure has a close connection to the continued fraction expansion of an irrational. If there were badly approximable functions the topology and distribution of power series would show phenomena similar to the topology and distribution of numbers. However, regardless whether there are badly approximable schlicht functions or not Theorems 1 and 2 provide a refined picture of the topology and distribution of schlicht functions as their application to asymptotic extremal problems (examples 1 and 2) shows.

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